Matching with Quotas

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September 26, 2016

Abstract

The paper characterizes the core of the many-to-one matching problem with quotas. The many-to-one matching problem with quotas may have an empty core, and there is no clear set of necessary and sufficient conditions that guarantee non-emptiness of the core. Usual sufficient conditions for non-emptiness of the core for matching problems cannot be applied for the problem with quotas. We introduce set strong substitutability of preferences, a refinement of strong substitutability. We show that if preferences are set strongly substitutable, then the core of many-to-one matching problem with quotas is non-empty. Moreover, we prove that in this case the core has a lattice structure with opposition of interests.

Keywords: Matching, Stability, Fixed Point, Quotas
JEL classification: D62; C78

1 Introduction

This paper deals with the many-to-one matching problem and its core. Given a set of students and colleges matching is an assignment of (groups) of students to each college. The matching problem was introduced by Gale and Shapley (1962). Gale and Shapley also proposed a mechanism for finding a stable solution in one-to-one and many-to-one cases. The many-to-one matching problem is well-studied and has various applications. For instance, the National Residency Matching Program (Roth and Peranson (1999)), Boston and New York public school matching procedures (Abdulkadiroğlu et al. (2005b) and Abdulkadiroğlu

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We are grateful to Alex Teytelboym, Marco Castillo, David Eil and César Martinelli for their helpful comments.

et al. (2005a)), United States Military Academy cadets assignments (Sönmez and Switzer (2013)), German, Hungarian, Spanish and Turkish College Admission mechanisms (Braun et al. (2010), Biró (2008), Romero-Medina (1998) and Balinski and Sönmez (1999)), and the Japan Residency Matching Program (Kamada and Kojima (2011)).

Most of many-to-one matching problems listed above come with capacity restrictions. For instance, rural hospitals require a minimum of doctors to operate and can not employ more than a certain maximum amount of doctors. Indeed, one can think of the use of capacity constraints as a way to secure the assignment of doctors across locales. Despite its clear practical implications, we know little about the core in many-to-one problems with quotas. The many-to-one matching problem is different from the one-to-one matching problem in various dimensions. The core of the many-to-one matching problem can be empty, unlike the core of the one-to-one problem. There are several sufficient conditions for non-emptiness of the core of the many-to-one matching problem (Roth (1985a) and Blair (1988)). These results hold for the problem with upper quotas (capacities), but cannot be directly applied for the problem with lower quotas. Moreover, to our knowledge there is no sufficient condition for non-emptiness of the core of many-to-one matching problem with both upper and lower quotas. We study the case in which each matching assignment is required to satisfy both lower and upper quotas, as in Biró et al. (2010).[1]

Hatfield and Milgrom (2005) show that substitutability[2] guarantees non-emptiness of the core for the matching with contracts. The importance of the substitutability condition for non-emptiness of the core has been shown for supply chain networks (Ostrovsky (2008)), ascending clock auctions (Milgrom and Strulovici (2009)), package auctions (Milgrom (2007)), many-to-one and many-to-many matching problems (Echenique and Oviedo (2004); Echenique and Oviedo (2006)). However, preferences in the problem with quotas cannot satisfy substitutability. This can be illustrated with a simple example. Let c be a college and s_1, s_2 be students. Assume that c prefers s_1 and s_2 together to s_1 and s_2 separately. Substitutability implies, that if some student is chosen from a given set of alternatives, then the same student would be chosen from a subset of original set of alternatives. Hence, the original preferences of c satisfy substitutability. However, if we impose the *lower quota of two* - the college can not operate having less than two students, then c would prefer to stay alone (choose nothing) from the sets of alternatives that contain only s_1 or s_2. This violates substitutability, even though preferences satisfy substitutability if we consider a problem without lower quotas. In the discussion we provide more formal discussion of this and provide a formal proof that preferences in the non-trivial many-to-one matching

[1] For the case in which these restrictions can be violated see Fragiadakis et al. (2016).

[2] Substitutability of preferences informally imply that all commodities or partners are substitutes. For the formal definition of substitutability see Section 4. For the expansive discussion on substitutability see Hatfield et al. (2016)

problem cannot satisfy substitutability.

We provide a condition on preferences that secures the existence of a matching in the core of the problem with quotas. Moreover, we show that if this condition holds, the core is a lattice with the opposition of the interests.[3] We prove the results using the fixed point approach from Echenique and Oviedo (2004) and Echenique and Oviedo (2006).

The remainder of this paper is organized as follows. In Section 2 we state the model of many-to-one matching with quotas. In Section 3 we show the results on non-emptiness of the core and the lattice structure of it. In Section 4 we discuss connection of the results to the previous literature and show that the conditions for non-emptiness of the core which were obtained before cannot be directly applied for the problem with quotas.

2 Preliminaries

A matching problem can be specified as a tuple $\Gamma = (N, \mathcal{M}, \mathcal{R})$, where N is the set of players, $\mathcal{M}$ is the set of all possible matchings and $\mathcal{R}$ is the preference profile.

Let us start from defining N, the set of players. For simplicity, we will call the two participating sides as students and colleges. We use $S = \{s_1, ..., s_n\}$ to denote the set of students and $C = \{c_1, ..., c_m\}$ to denote the set of colleges, and $N = S \cup C$.

2.1 Matching

Recall that we consider a matching problem with quotas. Therefore, to define $\mathcal{M}$ we need to define quotas first. Let the lower quotas be a function $\underline{q}(c) : C \to \mathbb{N}$ and the upper quotas be a function $\bar{q}(c) : C \to \mathbb{N}$ such that $\bar{q}(c) \geq \underline{q}(c)$. Every college has to get matched with at least $\underline{q}(c)$ partners and no more than $\bar{q}(c)$. Note that since we are considering many-to-one matching problem, students can be matched to either one or none of the colleges.

An assignment is a correspondence $\nu = (\nu_S, \nu_C)$, where $\nu_S : S \to C \cup \{\emptyset\}$ and $\nu_C : C \to 2^S$. A **prematching** is an assignment such that $s \in \nu_C(c)$ if and only if $c = \nu_S(s)$. Recall that students can be matched to no more than one college, while colleges can be matched to many students. Note that prematching does not require the colleges to satisfy quotas, i.e. in a prematching college can be matched to the amount of students that is less than lower quota or greater than upper quota. For the simplicity of further notation we can refer to the match of agent a as $\nu(a)$ that would be equal to $\nu_C(a)$ if $a \in C$ or $\nu_S(a)$ if $a \in S$.

Definition 1. *A prematching μ is said to be a* ***matching*** *if there are $S' \subseteq S$ and $C' = \bigcup_{s \in S'} \mu_S(s) \subseteq C$, such that*

[3]Under opposition of interests we mean conflicting interests as it was introduced by Roth (1985b).

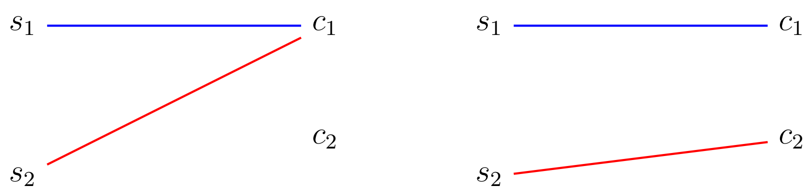
(i) for every $c \in C'$, $\underline{q}(c) \leq |\mu(c)| \leq \bar{q}(c)$ *and,*

(ii) for every $\bar{c} \in C \setminus C'$ $\mu_C(\bar{c}) = \emptyset$ *and,*

(iii) for every $\bar{s} \in S \setminus S'$ $\mu_S(\bar{s}) = \emptyset$

Note that presence of quotas requires that each college is either unmatched or matched to at least $\underline{q}(c)$ students. This assumption is due to the fact, that if a college (or a hospital) has less than a certain amount students than it cannot function. Therefore, the colleges that can not get at least $\underline{q}(c)$ prefer to close. Let $q = (\underline{q}, \bar{q})$ be vector of lower and upper quotas for all players, then we can denote by $\mathcal{M}(C, S, q)$ set of all matchings. The set can be characterized by the set of players (students and colleges) and the quotas.

Example: Let $C = \{c_1, c_2\}$, $S = \{s_1, s_2\}$ and $\underline{q}(c_1) = \underline{q}(c_2) = 2$ and $\bar{q}(c_1) = \bar{q}(c_2) = 3$



(a) Matching (b) Prematching, but not a matching

Figure 1: Illustration of difference between matching and prematching

Figure 1 illustrates the definition of a matching. Figure 1(a) shows an example of matching, where c_2 is unmatched, but s_1 and s_2 are matched with c_1. Therefore, c_1 fulfills its lower quota. Figure 1(b) shows an example of prematching which is not a matching. Both students are matched as well as both colleges, but neither college fulfills the lower quota.

2.2 Preferences

Now we need to define the preference profile $\mathcal{R}$, the set of preference relations. We assume that every agent $a \in C \cup S$ has a linear order[4] of preferences over possible matches. Denote by $R(s)$ the preference relation of student s and by $\tilde{R}(c)$ the preference relation of college c. A match X is said to be **acceptable** by agent $a \in C \cup S$ if $XR(a)\emptyset$, i.e. a match X is preferred to the stay-alone option. Let $R(c)$ be the *truncated preference relation*, such that any $X \neq \{\emptyset\}$ with $|X| < \underline{q}(a)$ or $|X| > \bar{q}(c)$ is unacceptable. Note that if $\tilde{R}(c)$ is a linear order, then $R(c)$ is a linear order as well, since it is a permutation of $\tilde{R}(c)$ that makes several sets of students unacceptable. Denote by $P(a)$ the strict part of preference relation of $R(a)$, for any $a \in C \cup S$.

Example: Let $C = \{c_1; c_2\}$, $S = \{s_1, s_2, s_3\}$ and $\underline{q}(c_1) = \underline{q}(c_2) = 2$ and $\bar{q}(c_1) = \bar{q}(c_2) = 2$.

[4] A linear order is a complete, anti-symmetric and transitive preference relation

$\tilde{R}(c_1) : \{s_1, s_2, s_3\}P(c_1)\{s_1, s_2\}P(c_1)\{s_1, s_3\}P(c_1)\{s_1\}P(c_1)\emptyset$ $R(c_1) : \{s_1, s_2\}P(c_1)\{s_1, s_3\}P(c_1)\emptyset$

(a) Original preference relation (b) Truncated preference relation

Figure 2: Illustration of truncated preference relation for colleges

Figure 2 illustrates the construction of truncated preference relation from the original one. Figure 2(a) shows the original preference relation over sets of alternatives. In this case the set $\{s_1, s_2, s_3\}$ is unacceptable, since it exceeds the capacity (upper quota) and the alternative $\{s_1\}$ is unacceptable since it would not allow to fulfill lower quota. Eliminating these alternatives we arrive to the truncated preference relation shown in Figure 2(b).

Let $Ch(X, R(a)) \subseteq X$ be the most preferred set of agent a.[5] Note that since we assume $R(a)$ to be a linear order, $Ch(X, R(a))$ is unique. So, $Ch(X, R(a))$ is the unique subset X' of X, such that $X'P(a)X''$ for any $X'' \subseteq X$. Note that for any $c \in C$ and $X \subseteq S$ if $|X| \leq \underline{q}(c)$ or $|X| \geq \bar{q}(c)$, then $Ch(X, R(c)) = \{\emptyset\}$.[6] Let us characterize some properties of a choice function generated by a linear order which we will use later:

- The choice function is **idempotent**. That is $Ch(Ch(X, R(a)), R(a)) = Ch(X, R(a))$.
- The choice function is **monotone**. That is for any $X \subseteq X'$, $Ch(X', R(a))\ R(a)\ Ch(X, R(a))$.

Definition 2. *An agent a's preference $R(a)$ satisfies **set strong substitutability** if for any X, X' such that $X'\tilde{R}(a)X$ and $x \in S$, such that $|X \cup \{x\}| \in [\underline{q}(a); \infty)$ and $|X' \cup \{x\}| \in [\underline{q}(a); \infty)$:*

$$x \in Ch(X' \cup \{x\}, R(a)) \ \Rightarrow \ x \in Ch(X \cup \{x\}, R(a))$$

Let us consider an example to illustrate what set strong substitutability is. Assume that there are three students $s_1, s_2, s_3 \in S$. If $X' = \{s_1, s_2\}R(c)\{s_1\} = X$, $S' = s_3$ and $Ch(X' \cup S', R(c)) = \{s_2, s_3\}$, then, $\{s_1\} = Ch(X \cup S', R(c))$ violates set strong substitutability. It implies that s_2 and s_3 are complementary - that is, they are chosen together, but not separately. This course likes s_2 and s_3 together better than s_1 alone, but if it can't have s_2 and s_3 together, then it would prefer s_1 alone rather than s_2 or s_3 alone. This sort of complementary is what set strong substitutability rules out.

A preference profile $\mathcal{R}$ is **set strongly substitutable** if $R(c)$ satisfies set strong substitutability for every college $c \in C$. Note that set strong substitutability only imposes additional restrictions on $R(c)$. $R(s)$ satisfies strong set substitutability by virtue of being a linear order over $C \cup \{\emptyset\}$. Note that requiring $\mathcal{R}$ to be set strongly subsistitutable is cruicial for the proof of our main result.

[5]In this case X is an arbitrary set that includes $\emptyset$ as an element.

[6]This statement is correct since we use the truncated preference relation of college $c \in C$.

2.3 Solution Concept

We define the solution concept for matching problem Γ as follows. A matching μ is said to be **individually rational** if $\mu(a) = Ch(\mu(a), R(a))$ for any $a \in C \cup S$. We consider the core as the solution concept since it is a general concept that can be defined for a game in abstract form.

Let us start from defining the core.[7] For this purpose we need to introduce the dominance relation.

Definition 3. *A matching ν **dominates** a matching μ if there is a $S' \neq \emptyset$, $S' \subseteq S$ and $C' = \bigcup_{s_i \in S'} \nu_S(s_i) \subseteq C$, such that*

(i) for every $a \in C' \cup S'$ $\nu(a)R(a)\mu(a)$, and

(ii) for some $a \in C' \cup S'$ $\nu(a)P(a)\mu(a)$.

Note that the dominance relation requires both ν and μ to be matchings - i.e., prematchings that satisfy quotas. We use the dominance in the Neumann et al. (1944) sense, that requires merging the standard concept of blocking coalition with the restriction on the set of matchings. This merging is necessary, since dominance as it is defined in the literature on matching without quotas allows a prematching to dominate a matching. We restrict the set of feasible coalitions to matchings only, therefore matchings cannot be dominated by prematchings that are not matchings. That is, if a course wants to leave the market and get better students on its own, it still has to fulfil lower quotas.

Definition 4. *The **core** ($\mathcal{C} \subseteq \mathcal{M}$) is a collection of individually rational matchings $\mu \in \mathcal{C}$, such that there is no individually rational matching $\mu' \in \mathcal{M}$ that dominates $\mu \in \mathcal{C}$.*

Note that if μ' dominates individually rational μ it does not necessarily imply the individual rationality of μ', since it may be the case that μ' assigns some unacceptable partners to some agents. However, if there is μ' that dominates individually rational μ then there always exists an individually rational μ'' that dominates μ. For instance, we can take all agents that are assigned unacceptable partners and assign them $\emptyset$, since these people are not in $C' \cup S'$. Therefore, changing partners assigned to these agents would not affect dominance.

3 Results

Let us now state the main result of the paper: a sufficient condition for non-emptyness of the core.

[7] The definition of the core we are using is sometimes called an individually rational core, since it constraints the set of dominant matchings to the individually rational matchings.

Theorem 1. *If the preference profile $\mathcal{R}$ is set strongly substitutable, then the core is not empty.*[8]

The proof has the following structure. We introduce an operator T and show that the set of its fixed points is a subset of the core of Γ. Second, we prove that T is a monotone operator over a complete lattice. This will allow us to apply Tarski's fixed point theorem to show that $\mathcal{E}$ is non-empty, and therefore the core is non-empty.

Therefore, proof of Theorem 1 requires the following Lemmata.

3.1 Lemmata

To prove Theorem 1 we need to introduce the following definitions.

Denote by $\mathcal{V}_S = (C \cup \{\emptyset\})^S$ the set of all assignments that students can get. Denote by $\mathcal{V}_C \subseteq (2^S)^C$ set of all assignments that satisfies colleges' quotas. Say that a pair $\nu = (\nu_S, \nu_C)$ with $\nu_S : S \to C \cup \{\emptyset\}$ and $\nu_C : C \to 2^S$ and $\nu_S \in \mathcal{V}_S$, $\nu_C \in \mathcal{V}_C$ is an **assignment**. Note that difference between matching and assignment is that an assignment does not have to be mutually consistent, i.e. if $s \in \nu_C(c)$ in assignment ν, it does not require that $c = \nu_S(s)$. Let $\mathcal{V} = \mathcal{V}_S \times \mathcal{V}_C$. For any assignment $\nu \in \mathcal{V}$ of agent a, either $\nu(a) = \emptyset$, or $\underline{q}(a) \leq |\nu(a)| \leq \bar{q}(a)$. Therefore, the required preference profile is defined only over the assignments that fill out quotas and the assignment to stay alone - that is, assignment to the empty set.

Now we move on to introducing the T operator. We use the T-operator from Echenique and Oviedo (2004).

Definition 5. *Let ν be an assignment, then*

- *Let $U(c,\nu) = \{s \in S : \ cR(s)\nu(s)\}$ for any $c \in C$*
- *Let $V(s,\nu) = \{c \in C : \ s \in Ch(\nu(c) \cup \{s\}, R(c))\}$ for any $s \in S$.*

Set $U(c,\nu)$ is the set of students that prefer c to the current assignment of s. Set $V(s,\nu)$ is the set of colleges that would include s into their most preferred set from $\nu_C(c) \cup \{s\}$.

Definition 6. *Now define $T : \mathcal{V} \to \mathcal{V}$ by*

$$(Tv)(a) = \begin{cases} Ch(U(a,\nu), R(a)) \ if \ a \in C \\ Ch(V(a,\nu), R(a)) \ if \ a \in S \end{cases}$$

for any $a \in C \cup S$.

[8]Theorem 1 and all following results do not require the preferences to be linear orders, all the results would hold if for every $c \in C$ the preference relation is a lattice.

Then T operator assigns to every $c \in C$ the most preferred set of students among those $s \in S$ that prefers c to the current assignment and to every $s \in S$ the most preferred college among those $c \in C$ that chooses s from some $\nu_C(c) \cup \{s\}$. An assignment ν is said to be a **fixed point** of T if $T\nu = \nu$. Denote by $\mathcal{E}$ the set of fixed points, i.e. $\mathcal{E} = \{\nu \in \mathcal{V} : \ \nu = T\nu\}$.

Hence, the **T-algorithm** is the procedure of iterating T starting at some assignment $\nu \in \mathcal{V}$. Note that operator T starts from an assignment and iterates by creating a new assignment that is not necessarily a prematching or a matching, but satisfies quotas. Hence, we need to show that if T-algorithm stops, then it stops at the matching.[9]

Lemma 1. *If $\nu \in \mathcal{E}$, then ν is an individually rational matching*

Proof of Lemma 1 is similar to the proof of Lemma 11.4 from Echenique and Oviedo (2006). Therefore, the proof is in the Appendix.

Lemma 2. *$\mathcal{E} = \mathcal{C}$.*

Lemma 2 is equivalent to the Corollary 4 from Echenique and Oviedo (2004). Therefore, proof is in the Appendix.

We have now shown that every fixed point of T is an element of the core. Now we need to show that T is a monotone operator over the lattice of assignments that satisfy quotas. To define the lattice over $\mathcal{V}$ we need to define a partial order over $\mathcal{V}$ with respect to which T is monotone.

Definition 7. *Define*

- $\leq_C$ *on* $\mathcal{V}_C$ *by* $\nu'_C \leq_C \nu_C$ *if for every* $c \in C$ $\nu_C(c) R(c) \nu'_C(c)$.
  The strict part of $\leq_C$ *is* $<_C$ *is* $\nu'_C <_C \nu_C$ *if* $\nu'_C \leq_C \nu_C$ *and* $\nu_C \neq \nu'_C$.
- $\leq_S$ *on* $\mathcal{V}_S$ *by* $\nu'_S \leq_S \nu_S$ *if for every* $s \in S$ $\nu_S(s) R(s_i) \nu'_S(s)$.
  The strict part of $\leq_S$ *is* $<_S$ *is* $\nu'_S <_S \nu_S$ *if* $\nu'_S \leq_S \nu_S$ *and* $\nu_S \neq \nu'_S$.
- $\leq_{CS}$ *on* $\mathcal{V}$ *by* $\nu' \leq_{CS} \nu$ *if* $\nu'_C \leq_C \nu_C$ *and* $\nu_S \leq_S \nu'_S$.
  The strict part of $\leq_{CS}$ *is* $<_{CS}$ *is* $\nu' <_C \nu$ *if* $\nu' \leq_{CS} \nu$ *and* $\nu \neq \nu'$.
- $\leq_{SC}$ *on* $\mathcal{V}$ *by* $\nu' \leq_{SC} \nu$ *if* $\nu \leq_{CS} \nu'$.

Note that $\leq_{CS}$ introduces the opposition of interests between colleges and students, since $\nu \leq_{CS} \nu'$ requires ν' to be more preferred by all colleges and less preferred by all students. And $\leq_{SC}$ is the reverse of the partial order $\leq_{CS}$. Let $\mathcal{V}' = \{\nu : \nu(a) R(a) \emptyset, \ \forall a \in C \cup S\}$, note that for any $\nu \in \mathcal{V}$, $T\nu \in \mathcal{V}'$.

[9] Since T-algorithm can stop only at the fixed point of T operator.

Lemma 3. *Let $\mathcal{R}$ be set strongly substitutable and $\nu, \nu' \in \mathcal{V}'$. Then, $\nu \leq_{CS} \nu'$ implies that for any $s \in S$ and $c \in C$: $U(c,\nu) \subseteq U(c,\nu')$ and $V(s,\nu') \subseteq V(s,\nu)$.*

Proof. ($V(s,\nu') \subseteq V(s,\nu)$). If $V(s,\nu') = \emptyset$, then the claim is trivially correct. Therefore, assume that $V(s,\nu) \neq \emptyset$, then there is $c \in V(s,\nu')$. By the definition of V: $s \in Ch(\nu'(c) \cup \{s\}, R(c))$, note that since $V(s,\nu')$ is non-empty then ν' has at least $\underline{q}(c)$ elements. Since $\nu \leq_{CS} \nu'$, then $\nu'(c)R(c)\nu(c)$, then by set strong substitutability $s \in Ch(\nu(c) \cup \{s\}, R(c))$. In this case to guarantee that $|\nu(c) \cup \{s\}| \geq \underline{q}(a)$, because ν' has at least $\underline{q}(c)$ elements, therefore, $\nu(c)R(c)\nu'(c)R(c)\emptyset$ cannot have less than $\underline{q}(c)$ elements, otherwise it is unacceptable and cannot be preferred to ν'. Hence $c \in V(s,\nu)$.

($U(c,\nu) \subseteq U(c,\nu')$). If $s \in U(c,\nu)$, then $cR(s)\nu(s)$. But $\nu(s)R(s)\nu'(s)$, hence $cR(s)\nu'(s)$. Therefore, $s \in U(c,\nu')$. □

Lemma 4. *If $\mathcal{R}$ is set strongly substitutable, then restricted operator $T|_{\mathcal{V}'}$ is a monotone map over $\mathcal{V}'$ endowed with $\leq_{CS}$ ($\leq_{SC}$).*

Proof of Lemma 4 is similar to the proof of Lemma 14.3 from Echenique and Oviedo (2006), therefore is moved to the Appendix. Now we can easily prove Theorem 1 using the Lemmata.

Proof of Theorem 1. Note that $T(\mathcal{V}) \subseteq \mathcal{V}'$, then restricted operator $T|_{\mathcal{V}'} : \mathcal{V}' \to \mathcal{V}'$ is a monotone operator (by Lemma 4) and $\mathcal{V}'$ is a complete lattice[10] and $\mathcal{E} \subseteq \mathcal{V}'$. Then by Tarski's fixed point theorem $(\mathcal{E}, \leq_{CS})$ is a non-empty complete lattice. From Lemma 1 we know that $\mathcal{E} \subseteq \mathcal{C}$, therefore core is non-empty. □

3.2 Lattice Structure of the Core

Further we can specify the structure of the core, using the Theorem 1. However, to define lattice it is necessary to establish a partial order. Further, we will be using order from Definition 7 but only over the set of matchings ($\mathcal{M}$) or the core ($\mathcal{C} \subseteq \mathcal{M}$).

Corollary 1. *If $\mathcal{R}$ is set strongly substitutable, then $(\mathcal{C}, \leq_{CS})$ and $(\mathcal{C}, \leq_{SC})$ are non-empty lattices.*

Note that in proof of Theorem 1 we have already shown that $\mathcal{E}$ is a non-empty lattice over $\mathcal{M}$ endowed with order $\leq_{CS}$ (or $\leq_{SC}$). And in the proof of Corollary 2 we shown that $\mathcal{C} = \mathcal{E}$.

Note that in the absence of set strong substitutability of $\mathcal{R}$ we can not guarantee the lattice structure of the core. Even if core is non-empty (T has at least one fixed point), we

[10] Since it is a finite lattice, and every finite lattice is complete.

still can not guarantee that $\mathcal{E}$ is a lattice, since we can not apply Tarki's fixed point theorem in the absence of set strong substitutability.

4 Discussion

The discussion covers the following three points. First, we show that the substitutability condition usually assumed to gather non-empty core in the matching problem without quotas cannot be applied for the non-trivial problem with quotas. Second, we discuss the relationship between the set strong substitutability and responsiveness of preferences. Third, we make a remark on the computational complexity of determining whether the core of the many-to-one matching problem with quotas is non-empty.

4.1 Substitutability

Echenique and Oviedo (2004) show that if preference profile is *substitutable*, then the core of many-to-one matching problem without quotas is non-empty. Let us show that this condition cannot be applied for the many-to-one problem with quotas.

Definition 8. *$R(c)$ for given $c \in C$ satisfies **substitutability** if for any $S' \subset S$, containing $s, \bar{s} \in S'$, $s \in Ch(S', R(c))$ implies $s \in Ch(S' \setminus \{\bar{s}\}, R(c))$.*

A preference profile $\mathcal{R}$ is **substitutable** if $R(c)$ satisfies substitutability for every college $c \in C$.

Lemma 5. *If there is a college $c \in C$, such that $\underline{q}(c) \geq 2$ and c has at least one acceptable set of students, then $\mathcal{R}$ is not substitutable.*

Lemma 5 shows that if there is at least one college quota at least two, then the college's truncated preference relation violates substitutability. Hence, the condition from Echenique and Oviedo (2004) can not be applied for the problem with quotas. However, if we consider a many-to-one matching problem with $\underline{q}(c) \leq 1$ for every $c \in C$, then set strong substitutability implies substitutability. In this sense set strong substitutability can be considered as a refinement of substitutability condition for the problem with quotas.

Example: Let $C = \{c_1\}$, $S = \{s_1, s_2\}$, $\underline{q}(c_1) = \bar{q}(c_1) = 2$.

$$\tilde{R}(c_1): \ \{s_1, s_2\} P(c_1) s_1 P(c_1) s_2$$
$$R(c_1): \ \{s_1, s_2\}$$

We can illustrate this using $R(c_1) : \{s_1, s_2\}$, the preference relation trivially satisfies set strong substitutability. And at the same time it violates substitutability, since $\{s_1, s_2\} =$

$Ch(\{s_1, s_2\}, R(c_1))$, then by substitutability $\{s_1\}$ should be equal to $Ch(\{s_1\}, R(c_1))$, but that is impossible, since $\{s_1\}$ violates the lower quota $\underline{q}(c_1) = 2$. At the same time $\mathcal{R}$ *satisfies set strong substitutability* and we can guarantee non-emptiness of the core using Theorem 1.

4.2 Responsiveness

Responsiveness of preferences originally introduced by Roth (1985a) is another sufficient condition for non-emptiness of the core in the many-to-one matching problem without quotas. Biró et al. (2010) show that it is sufficient for the non-emptiness of the core in the many-to-one matching problem with common lower quotas. Moreover, both set strong substitutability and responsiveness are deeply connected to the separability[11] of preferences.[12] However, neither condition is stronger than another. To show this, let us define responsiveness.

Definition 9. *$R(c)$ for given $c \in C$ satisfies* ***responsiveness*** *if for any $s, s' \subset S \cup \{\emptyset\}$, $sR(c)s'$ and $s \in S'$ implies $S'R(c)(S' \setminus \{s\}) \cup \{s'\}$.*

A preference profile $\mathcal{R}$ is **responsive** if $R(c)$ satisfies responsiveness for every college $c \in C$. Note that Biró et al. (2010) requires the original preference profiles ($\tilde{R}(c)$) to be responsive. If there is a college, such that its upper quota is less than number of acceptable students, then the preferences can not satisfy responsiveness. Take the most preferred set of students from the truncated preference relation of college c. Then according to the definition of responsiveness union of this set with any of acceptable students should be preferred by course c. Due to the constraint imposed by the upper quota, any such set would be unacceptable. That is a violation of responsiveness. If the lower quota for the college is greater than one, then responsiveness of preferences implies that the most preferred set of students should be $\emptyset$. Hence, if there is at least one acceptable set of students, then the preference relation does not satisfy responsiveness. Further, we consider the responsiveness as a property of the original preference relation and set strong substitutability as a property of the truncated preference relation.

Let us firstly show that there is a set strongly substitutable profile which is not responsive.

Example: Let $C = \{c_1\}$, $S = \{s_1, s_2, s_3\}$ and $\underline{q}(c_1) = 2$ and $\bar{q}(c_1) = 3$.

$$\tilde{R}(c_1) : \{s_1, s_2, s_3\}\tilde{P}(c_1)\{s_1, s_3\}\tilde{P}(c_1)\{s_1, s_2\}\tilde{P}(c_1)s_1\tilde{P}(c_1)s_2\tilde{P}(c_1)s_3$$
$$R(c_1) : \{s_1, s_2, s_3\}P(c_1)\{s_1, s_3\}P(c_1)\{s_1, s_2\}$$

[11] $R(c)$ for given $c \in C$ satisfies **separability** if for any set of students $S' \subseteq S$ and a student $s \in S$, $S' \cup sP(c)S' \setminus s$ if and only if $sP(c)\emptyset$.

[12] Separability implies both strong substitutability, the analog of set strong substitutability for the problem without lower quotas, and responsiveness of preferences.

The preference relation $\tilde{R}(c_1)$ does not satisfy responsiveness, since then $s_1\tilde{P}(c_1)s_3$ would imply that $\{s_1, s_2\}\tilde{P}(c_1)\{s_1, s_3\}$ which is not true. Note that in this case $R(c_1)$ does not satisfy responsiveness either. At the same time, the truncated preference relation $R(c_1)$ is set strongly substitutable.

Now let us provide an example of the preference relation that is responsive but not set strongly substitutable.

Example: Let $C = \{c_1\}$, $S = \{s_1, s_2, s_3, s_4, s_5\}$ and $\underline{q}(c_1) = 1$ and $\bar{q}(c_1) = 2$. And let the truncated preference profile to contain the following chains:

$$\ldots\{s_1, s_4\}\tilde{P}(c_1)\{s_1, s_5\}\tilde{P}(c_1)\{s_2, s_3\}\tilde{P}(c_1)\{s_2, s_5\}\ldots$$

and

$$\ldots s_1\tilde{P}(c_1)s_2\tilde{P}(c_1)s_3\tilde{P}(c_1)s_4\tilde{P}(c_1)s_5$$

$\tilde{R}(c_1)$ satisfies responsiveness, but $R(c_1)$ fails set strong substitutability. Let $X' = \{s_1, s_5\}$, $X = \{s_2, s_3\}$. Then $X'R(c_1)X$ and $s_4 \in Ch(X' \cup \{s_4\}, R(c_1)) = \{s_1, s_4\}$ and $s_4 \notin Ch(X \cup \{s_4\}, R(c_1)) = \{s_2, s_3\}$, which is a violation of set strong substitutability.

Therefore, set strong substitutability and responsiveness are alternative conditions that can be applied to guarantee non-emptiness of the core in many-to-one matching problem with quotas and neither condition is stronger than another.

4.3 Computational Complexity

Before we start a discussion of computational complexity let us recall the definition of T-algorithm. Say that a pair $\nu = (\nu_S, \nu_C)$ with $\nu_S : S \to C \cup \{\emptyset\}$ and $\nu_C : C \to 2^S$ and $\nu_S \in \mathcal{V}_S$, $\nu_C \in \mathcal{V}_C$ is an **assignment**. Note that difference between matching and assignment is that an assignment does not have to be mutually consistent, i.e. if $s \in \nu_C(c)$ in assignment ν, it does not require that $c = \nu_S(s)$. Recall the set of all assignments is denoted by $\mathcal{V}$.

Definition 5. *Let ν be an assignment, then*

- *Let $U(c, \nu) = \{s \in S : \ cR(s)\nu(s)\}$ for any $c \in C$*
- *Let $V(s, \nu) = \{c \in C : \ s \in Ch(\nu(c) \cup \{s\}, R(c))\}$ for any $s \in S$.*

Set $U(c, \nu)$ is the set of students that prefer c to the current assignment of s. Set $V(s, \nu)$ is the set of colleges that would include s into their most preferred set from $\nu_C(c) \cup S'$ for some S' that includes s.

Definition 6. *Now define* $T : \mathcal{V} \to \mathcal{V}$ *by*

$$(Tv)(a) = \begin{cases} Ch(U(a,\nu), R(a)) \; if \;\; a \in C \\ Ch(V(a,\nu), R(a)) \; if \;\; a \in S \end{cases}$$

for any $a \in C \cup S$.

Hence, the T**-algorithm** is the procedure of iterating T starting at some assignment $\nu \in \mathcal{V}$. Note that at the intermediate steps T-algorithm returns assignments, which are not necessary matchings, but as we shown above, if T-algorithm stops it returns a matching.

Example: Let $C = \{c_1, c_2\}$, $S = \{s_1, s_2\}$, $\underline{q}(c_1) = \underline{q}(c_2) = \bar{q}(c_1) = \bar{q}(c_2) = 2$.

$$\begin{aligned} R(s_1) &: \; c_1 P(s_1) c_2 \\ R(s_2) &: \; c_2 P(s_2) c_1 \\ \tilde{R}(c_1) &: \; \{s_1, s_2\} P(c_1) s_1 P(c_1) s_2 \\ \tilde{R}(c_2) &: \; \{s_1, s_2\} P(c_2) s_2 P(c_2) s_1 \\ R(c_1) &: \; \{s_1, s_2\} \\ R(c_2) &: \; \{s_1, s_2\} \end{aligned}$$

The problem with quotas has two elements in the core: μ^{Q1}: $\mu^{Q1}(c_1) = \{s_1, s_2\}$, $\mu^{Q1}(c_2) = \emptyset$; and μ^{Q2}: $\mu^{Q2}(c_1) = \emptyset$, $\mu^{Q2}(c_2) = \{s_1, s_2\}$. However, in this case the core element can not be found in the polynomial time, despite the set strong substitability of preferences. The T-algorithm cycles starting from every element that is not in the core. It is necessary to start the algorithm from any possible initial point (any element of $\mathcal{V}$) to determine whether there are any elements in the core, even though we know that the core is non-empty. This is consistent with the finding from Biró et al. (2010) that determining whether many-to-one matching problem has a non-empty core is an NP problem.

It may be that there is some compromise between how restrictive a solution is and how fast it can be reached. Perhaps finding a core that is easily calculated would require uselessly implausible restrictions on preferences. If there is such a trade-off, then many algorithms could be useful to fill out the frontier of possibilities. A larger menu would be more likely to provide the right mix of assumptions and computational requirements for each application.

Appendix: Proofs

Appendix A: Proof of Theorem 1

Lemma 1. *If* $\nu \in \mathcal{E}$, *then* ν *is an individually rational matching*

Proof. Let $\nu = (\nu_C, \nu_S) \in \mathcal{E}$. We first show that ν is a prematching, and then we show that it respects quotas - that is, ν is a matching. Recall that ν is an assignment since it is element of $\mathcal{V}$. Hence we need to show that $s \in \nu_C(c)$ if and only if $\nu_S(s) = c$.

$(s \in \nu_C(c) \;\Rightarrow\; \nu_S(s) = c)$
Fix $s \in \nu_C(c)$. Then $\nu \in \mathcal{E}$ implies that $\nu(c) = (T\nu)(c) = Ch(U(c,\nu), R(c)) \;\Rightarrow\; s \in U(c,\nu)$. By definition of $U(c,\nu)$: $cR(s)\nu(s)$. Then $Ch(\nu_C(c), R(c)) = Ch((T\nu)(c), R(c)) = Ch(Ch(U(c,\nu), R(c)), R(c))$. Recall that choice function is idempotent, hence, $Ch(Ch(U(c,\nu), R(c)), R(c)) = Ch(U(c,\nu), R(c))$ and $Ch(U(c,\nu), R(c)) = \nu_C(c)$. Then, $Ch(U(c,\nu), R(c)) = \nu_C(c)$. Hence, $Ch(\nu_C(c), R(c)) = \nu_C(c)$, i.e. $\nu_C(c)$ is individually rational.

Since $s \in \nu_C(c)$, therefore $Ch(\nu_C(c), R(c)) = Ch(\nu_C(c) \cup \{s\}, R(c))$. Then $c \in V(s,\nu)$. At the same time $\nu_S(s) = (T\nu)(s) = Ch(V(s,\nu), R(s))$, therefore, $\nu_S(s) \subseteq V(s,\nu)$. Then, $Ch(\nu_S(s) \cup \{c\}, R(s)) \subseteq \nu_S(s) \cup \{c\} \subseteq V(s,\nu)$. Since $\nu_S(s)$ is chosen from $V(s,\nu)$ $\nu_S(s)R(s)Ch(\nu_S(s) \cup \{c\}, R(s))$, hence $\nu_S(s)R(s)cR(s)\nu_S(s)$. Therefore, $\nu_S(s) = c$.

$(c = \nu_S(s) \;\Rightarrow\; s \in \nu_C(c))$
Fix $c = \nu_S(s)$. Then $\nu \in \mathcal{E}$ implies $\nu(s) = (T\nu)(s) = Ch(V(s,\nu), R(s))$. Hence, there is $s \in Ch(\nu_C(c) \cup \{s\}, R)$. $Ch(\nu_S(s), R(s)) = Ch((T\nu)(s), R(s)) = Ch(Ch(V(s,\nu), R(s)), R(s))$. Recall that choice function is idempotent, hence, $Ch(\nu_S(s), R(s)) = Ch(Ch(V(s,\nu), R(s)), R(s))$ and $Ch(V(s,\nu), R(s)) = \nu_S(s)$. Then, $Ch(V(s,\nu), R(s)) = \nu_S(s)$. Hence, $Ch(\nu_S(s), R(s)) = \nu_S(s)$, i.e. $\nu_S(s)$ is individually rational

Since $c = \nu_S(s)$, then $s \in U(c,\nu)$ and $\nu_C(c) \subseteq U(c,\nu)$ since ν is a fixed point. But $\nu_C(c) = (T\nu_C)(c) = Ch(U(c, \nu_C(c)), R(c))$ then $\nu_C(c) \subseteq U(c, \nu_C(c))$. Then, $Ch(\nu_C(c) \cup \{s\}, R(c)) \subseteq \nu_C(c) \cup \{s\} \subseteq U$. Hence, $Ch(\nu_C(c) \cup \{s\}, R(c))R(c)\nu_C(c)R(c)Ch(\nu_C(c) \cup \{s\}, R(c))$.

Therefore, ν is prematching. Now let us show that ν respects quotas, i.e. that if $\nu \in \mathcal{V}$, then $T\nu \in \mathcal{V}$. Recall that $\nu_C(c) = Ch(U(c,\nu), R(c))$, and by construction of $R(c)$ the smallest set that is preferable to $\emptyset$ is a set that contains at least $\underline{q}(a)$ elements. From the individual rationality of $\nu_C(c)$ ($\nu_C(c) = Ch(\nu_C(c), R(c))$) we can infer that if $\nu_C(c) \neq \emptyset$ is non-empty, then it contains at least $\underline{q}(a)$ elements Then $\nu_C(c)R\emptyset$, i.e. ν respects quotas and is a matching. □

Lemma 2. $\mathcal{E} = \mathcal{C}$.

Proof. ($\mathcal{E} \subseteq \mathcal{C}$). From Lemma 1 we know that $\nu = T\nu$ is individually rational matching. To prove that ν is in the core we assume that there is matching ν' that dominates ν and construct a contradiction. By assumption, there is $S' \neq \emptyset$ and $C' = \bigcup_{s \in S'} \nu_S(s) \subseteq C$, such that $\nu'(a)R(a)\nu(a)$ for every $a \in C' \cup S'$, and $\nu'(a)P(a)\nu(a)$ for some $a \in C' \cup S'$.

Then, without loss of generality assume there is $\bar{c} \in C'$, such that $\nu'(\bar{c})P(\bar{c})\nu(\bar{c})$. By individual rationality we know that $Ch(\nu(\bar{c}) \cup \nu'(\bar{c}), P(\bar{c})) \nsubseteq \nu(\bar{c})$. Let $\bar{s} \in \nu'(\bar{c}) \setminus \nu(\bar{c})$, then

$\bar{s} \in Ch(\nu(\bar{c}) \cup \nu'(\bar{c}) \cup \{\bar{s}\}, P(\bar{c})) = Ch(\nu(\bar{c}) \cup (\nu'(\bar{c}) \cup \{\bar{s}\}), P(\bar{c}))$. Then $\bar{c} \in V(\bar{s}, \nu)$ by definition of $V(\bar{s}, \nu)$.

Since $\bar{s} \in \nu'(\bar{c}) \backslash \nu(\bar{c})$, $\bar{s} \in S'$ and $\nu'(\bar{s}) \neq \nu(\bar{s})$. Hence, by antisymmetry of $R(\bar{s})$, $\bar{c}P(\bar{s})\nu(\bar{s})$. At the same time $\nu(\bar{s}) \cup \bar{c} \subseteq V(\bar{s}, \nu)$. Since ν is a fixed point, $\nu(\bar{s}) = Ch(V(\bar{s}, \nu), R(\bar{c}))R\bar{c}$. Hence, $\nu(\bar{s})P(\bar{s})\nu(\bar{s})$ that is a contradiction.

($\mathcal{C} \subseteq \mathcal{E}$) To prove that $\mathcal{C} \subseteq \mathcal{E}$ assume to the contrary that is $\mu \in \mathcal{C}$ and $\mu \notin \mathcal{E}$. Fix $c \in C$, such that $\mu(c) \neq Ch(U(c, \mu), R(c))$, then let $\mu'(c) = Ch(U(c, \mu), R(c))$. Hence $\mu'(c)P(c)\mu(c)$, because $\mu(c) \subseteq U(c, \mu)$. Now let $\mu'(s) = c$ for every $s \in \mu'(c)$. Let $\forall \bar{s} \in S \setminus \mu'(c)$ $\mu'(\bar{s}) = \emptyset$ and $\forall \bar{c} \in C \setminus \{c\}$ $\mu'(c) = \emptyset$.

Now let us show that μ' is an individually rational matching that dominates μ. Note that μ is a matching by construction. $\mu'(c)$ satisfies quotas, since it is obtained using the truncated preference relation, and for the rest of colleges $\mu'(\bar{c}) = \emptyset$. μ' is individually rational, since $\mu'(a) = \emptyset$ for every agent $a \in (C \cup S) \setminus (\mu(c) \cup c)$ and $\emptyset$ would be surely chosen from the set of alternatives that contains $\emptyset$ only. For c, matching μ' is individually rational by construction (as a chosen set from $U(c, \mu)$ and for every $s \in \mu'(c)$ it is individually rational since $cR(s)\mu(s)R(s)\emptyset$, since μ is individually rational. To show that μ' dominates μ denote by $C' = \{c\}$ and by $S' = \mu'(c)$. Then for every $s \in S'$, $\mu'(s)R(s)\mu(s)$ and for c, $\mu'(c)P(c)\mu(c)$.

Therefore, $\mu \notin \mathcal{C}$ that contradicts the assumption we made in the beginning. Hence, we have shown that $\mathcal{C} \subseteq \mathcal{E}$. This implies that $\mathcal{C} = \mathcal{E}$ and concludes the proof. □

Lemma 4. *If $\mathcal{R}$ is set strongly substitutable, then restricted operator $T|_{\mathcal{V}'}$ is a monotone map over $\mathcal{V}'$ endowed with $\leq_{CS}$ ($\leq_{SC}$).*

Proof. We need to prove that whenever $\nu \leq_{CS} \nu'$, then $T\nu \leq_{CS} T\nu'$. Let $\nu \leq_{CS} \nu'$, fix $c \in C$ and $s \in S$. The proof consists of two parts, we need to show that $(T\nu')(c)R(c)(T\nu)(c)$ and $(T\nu)(s)R(s)(T\nu')(s)$.

($(T\nu')(c)R(c)(T\nu)(c)$).
From Lemma 3 we know that $U(c, \nu) \subseteq U(c, \nu')$. Then we can show that:

$$Ch(U(c, \nu'), R(c)) = Ch([Ch(U(c, \nu'), R(c)) \cup Ch(U(c, \nu), R(c))], R(c))$$

Let $X \subseteq Ch(U(c, \nu'), R(c)) \cup Ch(U(c, \nu), R(c))$, then $X \subseteq U(c, \nu') \cup U(c, \nu) = U(c, \nu')$. Therefore, $Ch(U(c, \nu'), R(c))R(c)X$ and at the same time $Ch(U(c, \nu'), R(c)) \subseteq Ch([Ch(U(c, \nu'), R(c)) \cup Ch(U(c, \nu), R(c))], R(c))$. Therefore we shown that $Ch(U(c, \nu'), R(c)) = Ch([Ch(U(c, \nu'), R(c)) \cup Ch(U(c, \nu), R(c))], R(c))$. Note that $(T\nu')(c) = Ch(U(c, \nu'), R(c))$ and $(T\nu)(c) = Ch(U(c, \nu), R(c))$, then:

$$(T\nu')(c) = Ch([(T\nu')(c) \cup (T\nu)(c)], R(c))$$

Hence, $(T\nu')(c)R(c)(T\nu)(c)$.

$((T\nu)(s)R(s)(T\nu')(s))$.

From Lemma 3 we know that $V(c,\nu') \subseteq V(c,\nu)$. Then, $(T\nu)(s) = Ch(V(c,\nu), R(s))\ R(s)\ Ch(V(c,\nu'), R(s)) = T(\nu')(s)$.

Therefore, $T\nu \leq_{CS} T\nu'$, by the definition of $\leq_{CS}$. Similar proof can be conducted for $\leq_{SC}$. □

Appendix B: Proof of Lemma 5

Lemma 5. *If there is a college $c \in C$, such that $\underline{q}(c) \geq 2$ and c has at least one acceptable set of students, then $\mathcal{R}$ is not substitutable.*

Proof. If there is $c \in C$ such that $\tilde{R}(c)$ does not satisfy substitutability, then $R(c)$ does not satisfy substitutability. Therefore, assume that for every $c \in C$ $\tilde{R}(c)$ satisfies substitutability. The fix $c \in C$ and $s \in Ch(S^0, R(c)$ for some $S^0 \subseteq S$ such that $\underline{q}(c) \leq |S^0| \leq \bar{q}(c)$. Take $\bar{s}^0 \in S^0 \setminus \{s\}$, and consider $S^1 = S^0 \setminus \{\bar{s}^0\}$, then by substitutability of $R(c)$ $s \in Ch(S^1, R(c))$. We can repeat this procedure until $2 \leq |S^k| < \underline{q}(c)$, then $s \notin Ch(S^k \setminus \{\bar{s}^k\}, R(c))$, that violates substitutability. □

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